# Towards a 3D reduction of the N-body Bethe-Salpeter equation.

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## 1 Introduction.

The Bethe-Salpeter equation is the usual tool for computing bound states of relativistic particles. The principal difficulty of this equation comes from the presence of N-1 (for N particles) unphysical degrees of freedom: the relative time-energy degree of freedom. In the two-body problem, the relative energy is usually eliminated by replacing the free two-body propagator by an expression combining a delta fixing the relative energy and a 3D propagator. The exact equivalence (in what concerns the physically measurable quantities of the pure two-fermion problem) with the original Bethe-Salpeter equation can be obtained by recuperating the difference with the original free propagator in a series of correction terms to the 3D potential. It is not possible to generalize this constraining propagator-based reduction method to three or more particles, because of the unconnectedness of the two-body terms of the Bethe-Salpeter kernel, which are in fact the more important terms and often the only ones to be considered.

A less often used 3D reduction method is based on the replacement of the Bethe-Salpeter kernel by an "instantaneous" (i.e. independent of the relative energy) approximation (kernel-based reduction). In this case, the resulting 3D potential is not manifestly symmetric (i.e. hermitian when the total energy on which it depends is treated as a parameter). In the two-fermion problem, we obtained a symmetric 3D potential by performing a supplementary series expansion at the 3D level and combining it with the first 3D reducing expansion. We found that the starting instantaneous approximation of the Bethe-Salpeter kernel disappears from the final 3D potential. In fact, this potential can be obtained directly by a new integrating propagator-based reduction method, in which the relative energy is integrated on, instead of being fixed by a $\delta-$fonction (or constraint).

This integrating propagator-based reduction can easily be generalized to a system of N particles, consisting in any mixing of bosons and fermions [1, 2].

## 2 Inhomogeneous and homogeneous Bethe-Salpeter equations for 2 fermions:

$$G = G^0 + G^0 K G, \qquad \Phi = G^0 K \Phi$$

| | |
|---|---|
| $\Phi$ : | Bethe-Salpeter amplitude |
| $K$ : | Bethe-Salpeter kernel |
| | (must give $G$ via the inhomogeneous equation) |
| $G \equiv G^0 + G^0 T G^0$ : | Full propagator (Feynman graphs) |
| $G^0$ : | Free propagator: |

$$G^0 = G^0_1 G^0_2, \qquad G^0_i = \frac{1}{\gamma_i \cdot p_i \, - \, m_i \, + i\epsilon} = \frac{1}{p_{i0} - h_i + i\epsilon h_i}\, \beta_i$$

$$h_i = \vec{\alpha}_i \,.\vec{p}_i + \beta_i \, m_i \qquad (i = 1, 2)$$

The self-energy parts of the propagator were transferred to the kernel. We shall neglect them here for simplicity. $K$ is then the sum of the irreducible two-fermion Feynman graphs.

Notations for the following:

$$P = p_1 + p_2 \ , \qquad p = \frac{1}{2}(p_1 - p_2)$$

$$E = E_1 + E_2, \quad E_i = \sqrt{h_i^2} = (\vec{p}_i^{\,2} + m_i^2)^{\frac{1}{2}}.$$

$$\Lambda^+ = \Lambda_1^+ \Lambda_2^+, \qquad \Lambda_i^+ = \frac{E_i + h_i}{2E_i}, \qquad \beta = \beta_1 \beta_2$$

# 3 3D reduction by expansion around a positive-energy instantaneous approximation of K.

Write $K = K^0 + K^R$ with $K^0 = \Lambda^+ \beta K^0 \Lambda^+$ (positive-energy) and $K^0(p_0', p_0)$ independent of $p_0', p_0$ (instantaneous). The Bethe-Salpeter equation becomes

$$\Phi \, = \, G^0 K^0 \Phi \, + \, G^0 K^R \Phi \qquad \rightarrow$$

$$\Phi \, = \, (1 - G^0 K^R)^{-1} G^0 K^0 \Phi \qquad \rightarrow \qquad \Phi \, = \, (G^0 + G^{KR}) K^0 \Phi$$

with

$$G^{KR} \, = \, G^0 K^R (1 - G^0 K^R)^{-1} G^0.$$

Integrate with respect to $p_0'$ and apply $\Lambda^+$ $\rightarrow$ 3D equation:

$$\psi \, = \, (\, g^0 + g^{KR} \,) \, V^0 \, \psi$$

with

$$\Lambda^+ \int dp_0 G^0(p_0) \, = \, -2i\pi \Lambda^+ \, g^0 \, \beta, \qquad g^0 \, = \, \frac{1}{P_0 - E + i\epsilon}$$

$$\psi \, = \, \Lambda^+ \int dp_0 \, \Phi(p_0) \, , \qquad V^0 \, = \, -2i\pi \, \beta \, K^0 \, ,$$

$$g^{KR} \, = \, \frac{-1}{2i\pi} \, \Lambda^+ \int dp_0' dp_0 \, G^{KR}(p_0', p_0) \, \beta \, \Lambda^+.$$

# 4 Render the potential symmetric.

The 3D potential $(g^0)^{-1}(\, g^0 + g^{KR} \,) \, V^0$ is not symmetric. In Phillips and Wallace's method [3], one computes $K^0$ in order to make $g^{KR}$ vanish. Here, we shall write

$$g^{KR} = g^0 \, T^{KR} g^0$$

$$\rightarrow \qquad \psi = (1 + g^0 T^{KR}) \, g^0 V^0 \psi \qquad \rightarrow \qquad (1 + g^0 T^{KR})^{-1} \psi \, = \, g^0 V^0 \psi$$

$$\rightarrow \qquad \psi \, = \, \left[ \, g^0 V^0 + 1 - (1 + g^0 T^{KR})^{-1} \, \right] \psi \qquad \rightarrow \qquad \psi \, = \, g^0 \, V \, \psi$$

with

$$V = V^0 + T^{KR} (1 + g^0 T^{KR})^{-1}.$$

This potential $V$ is now symmetric.

## 5 Expand $T^{KR}$ and recombine the series.

$$T^{KR} \; = < K^R (1 - G^0 K^R)^{-1} >$$

with

$$< A > = \frac{1}{-2i\pi} \, \Lambda^+ (g^0)^{-1} \int dp'_0 dp_0 \, G^0(p'_0) A(p'_0, p_0) G^0(p_0) \, \beta \, \Lambda^+ (g^0)^{-1}.$$

This leads to

$$\begin{aligned} V \; &= < K^0 > + < K^R (1 - G^0 K^R)^{-1} > (1 + g^0 < K^R (1 - G^0 K^R)^{-1} >)^{-1} \\ &= < K^0 + K^R (1 - G^0 K^R)^{-1} (1 + > g^0 < K^R (1 - G^0 K^R)^{-1})^{-1} > \\ &= < K^0 + K^R (1 - G^0 K^R + > g^0 < K^R)^{-1} > = < K^0 + K^R (1 - G^R K^R)^{-1} > \end{aligned}$$

with the definitions

$$G^R \; = \; G^0 - G^I, \qquad G^I \; = \; > g^0 < .$$

Less formally:

$$G^0(p'_0, p_0) \; = \; G^0(p_0) \, \delta(p'_0 - p_0), \qquad G^I(p'_0, p_0) \; = \; G^0(p'_0) \, \beta \, \frac{\Lambda^+}{-2i\pi \, g^0} \, G^0(p_0).$$

but

$$\begin{aligned} K^0 G^R = G^R K^0 = 0 \; &\rightarrow \; K^R (1 - G^R K^R)^{-1} = -K^0 + K (1 - G^R K)^{-1} \; \rightarrow \\ V \; &= < \; K \, (1 - G^R K \,)^{-1} \; > = < K > + < K G^R K > + \cdots \\ &= < K > + \{ < K G^0 K > - < K > g^0 < K > \} + \cdots \end{aligned}$$

In the relative-energy integrals, $-G^I$ cancels the leading term coming from $G^0$.
Good surprise: $V$ does not depend on the initial choice of $K^0$ anymore.

## 6 We made in fact an integrating propagator-based reduction.

Our final 3D equation could also be obtained directly from the Bethe-Salpeter equation by performing an expansion around an approximation $G^I$ of the propagator $G^0$ ($\rightarrow$ integrating propagator-based reduction instead of the constraining propagator-based reduction using $\delta-$functions).

## 7 We could start with the equal-times retarded propagator.

Following [4] ([5] in the three-body case), we could also start by taking the retarded part of the full propagator at equal times. In momentum space, it is

$$g \; = \; g^0 \; + \; g^0 < T > g^0.$$

The corresponding 3D potential is

$$V \; = < T > (1 + g^0 < T >)^{-1}.$$

Writing then the expansion $T = K(1 - G^0 K)^{-1}$ and recombining the series gives $V = < K \, (1 - G^R K \,)^{-1} >$ again. Note that $T$ and $< T >$ are both proportional to the physical scattering amplitude when the initial and final fermions are on their positive-energy mass shell.

# 8 Generalization to systems of N particles.

Our 3D reduction method (as established in section 5 or section 6's way) can be easily generalized to systems consisting in any number of fermions and/or bosons.

Here we shall consider only the case of N fermions. The writing of the Bethe-Salpeter equation and of the final 3D equation remains the same:

$$\Phi \,=\, G^0 K\, \Phi \qquad \rightarrow \qquad \psi \,=\, g^0\, V\, \psi$$

$$V \;=< K(1-G^R K)^{-1} >, \qquad G^R \,=\, G^0 - \;\; > g^0 <, \qquad g^0 \,=\, \frac{1}{P_0 - E + i\epsilon}$$

with a trivial generalization of some notations:

$$\Lambda^+ = \Lambda_1^+ \cdots \Lambda_N^+, \qquad \beta = \beta_1 \cdots \beta_N,$$

$$P_0 = p_{01} + \cdots + p_{0N} \qquad E = E_1 + \cdots + E_N$$

$$< A >= \frac{1}{(-2i\pi)^{N-1}}\, \Lambda^+ (g^0)^{-1} \int dp_0' dp_0\, G^0(p_0') A(p_0', p_0) G^0(p_0)\, \beta\, \Lambda^+ (g^0)^{-1}$$

$$dp_0 \,=\, dp_{01} \cdots dp_{0N}\, \delta\, (\, p_{01} + \cdots + p_{0N} \,-\, P_0\,).$$

Expressions of $K$ for $N \geq 3$ :

$N = 3$ :

$$K = K_{12}(G_{03})^{-1} + K_{23}(G_{01})^{-1} + K_{31}(G_{02})^{-1} + K_{123}.$$

$N = 4$ :

$$\begin{aligned} K \;=\; & K_{12,34} \,+\, K_{13,24} \,+\, K_{14,23} \\ & +\, K_{123}\,(G_4^0\,)^{-1} \,+\, K_{124}\,(G_3^0\,)^{-1} \,+\, K_{134}\,(G_2^0\,)^{-1} \,+\, K_{234}\,(G_1^0\,)^{-1} \\ & +\, K_{1234}\,, \end{aligned}$$

with

$$K_{12,34} \,=\, K_{12}\,(G_3^0\, G_4^0\,)^{-1} \,+\, K_{34}\,(G_1^0\, G_2^0\,)^{-1} \,-\, K_{12}\, K_{34}, \qquad etc...$$

The counter-term $K_{12}\, K_{34}$ cancels the double-countings which would come from the fact that two graphs containing respectively $K_{12}\, K_{34}$ and $K_{34}\, K_{12}$ in the expansion of $G$ must be taken only once [6, 7, 8, 2].

$N \geq 5$ : Very complicated. For $N \geq 5$ (perhaps even for $N = 4$) we suggest to bypass the Bethe-Salpeter equation by writing $V = < T > (1 + g^0 < T >)^{-1}$ without expanding $T$ in terms of $K$, and sorting the contributing graphs by increasing number of vertexes.

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